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Semiclassical strings on $AdS_5 \times S^5/\mathbb{Z}_M$ and operators in orbifold field theories

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Abstract: We show agreements, at one-loop level of field theory, between energies of semiclassical string states on $AdS_5 \times S^5/\mathbb{Z}_M$ and anomalous dimensions of operators in $\mathcal{N} = 0, 1, 2$ orbifold field theories originating from $\mathcal{N} = 4$ SYM. On field theory side, one-loop anomalous dimension matrices can be regarded as hamiltonians of spin chains with twisted boundary conditions. These are solvable by Bethe ansatz. On string side, twisted sectors emerge and we obtain some string configurations in twisted sectors. In SU(2) subsectors, we compare anomalous dimensions with string energies and see agreements. We also see agreements between sigma models of both sides in SU(2) and SU(3) subsectors.

Keywords: 1/N Expansion, AdS-CFT and dS-CFT Correspondence, Bethe Ansatz.

Contents

1. Introduction

Recently, in context of AdS/CFT, energies of semiclassical string states on $AdS_5 \times S^5$ have successfully compared with anomalous dimensions of operators in planar $\mathcal{N} = 4$ Super Yang-Mills theory. In [1], it was shown that energy of semiclassical string states with large angular momentum in AdS_5 agrees qualitatively with anomalous dimension of twist two operators of $\mathcal{N} = 4$ SYM. On the string side, the analysis of semiclassical string states was generalized to cases of strings having angular momentum in S^5 in [2, 3]. Then, some kind of semiclassical string states with several large angular momentums in S^5 (and AdS_5) have been found to have energies which have analytic expansion in λ/J^2 (λ : 't Hooft coupling, J: angular momentum) [4]–[8].

On the other hand, it was shown that one-loop anomalous dimension matrix of operators consisting only of scalars in $\mathcal{N} = 4$ SYM is identical to a hamiltonian of SO(6) spin chain which is solvable by Bethe ansatz [9]. This was generalized to $\mathrm{SU}(2,2|4)$ super spin chain containing full sector of operators [11, 10]. Interestingly, integrability is likely to hold at higher loops [22]–[24]. Bethe ansatz including higher loops were investigated in [25]–[27].

The Bethe equations of spin chain are difficult to solve in general. In large number of sites limit, these equations are transformed to integral equations and can be solved. Then, the anomalous dimensions of the CFT operators can be obtained and these agree quantitatively with the energies of the semiclassical strings in SU(2) subsector at one-loop level [12]–[14]. Agreement at higher loops was investigated in [25, 26]. This agreement holds at 2-loop level but there is discrepancy between string and gauge sides at 3-loop level. Some discussions of the origin of this phenomena were done in these papers.

Moreover, an agreement between string and gauge sides was confirmed in the sigma model level [15]–[20] and general solution level [21]. Other interesting works concerning with semiclassical strings, anomalous dimensions and spin chains were done in many papers [38]–[45].

It is important to generalize the arguments of these agreements or integrabilities to other models. There are several simple ways of constructing these models; orbifolding [28] or deforming [29, 30] the $\mathcal{N}=4$ case, adding D-branes(open strings) [31]–[34], etc. (For other models see also [46]–[51].) In [28], integrabilities of orbifold field theories and their deformations from conformal fixed point were investigated purely in field theory point of view.

In this paper, we consider models obtained by orbifolding the $\mathcal{N}=4$ case. In untwisted sector, a story is almost the same as the original case. The matter in orbifold cases is whether boundary condition of twisted sector is compatible with integrability or not. If boundary condition does not break integrability, it is expected that, using this integrability, we can confirm agreements of string and gauge sides even for twisted sector. Indeed, this is the case and what we will do in this paper.

We analyze $\mathbb{C}^3/\mathbb{Z}_M$ orbifold cases at one-loop level of field theory. Our analysis is applied to $\mathcal{N}=0,1,2$ cases. We show integrabilities of one-loop anomalous dimension matrix in these models. Interestingly, we find that the integrabilities hold even in sectors corresponding to broken global symmetries. These sectors are described by solvable spin chains with twisted boundary conditions. Then, solving Bethe equations, we show agreements between energies of string states and anomalous dimensions of CFT operators in SU(2) subsectors. We also show agreements between sigma models of string and gauge sides in SU(2) and SU(3) subsectors.

Our paper is organized as follows. In section 2, we consider SU(2) subsectors of $\mathcal{N}=2$ cases. On string side, we construct 2-spin circular strings on $AdS_5 \times S^5/\mathbb{Z}_M$. On gauge side, we see integrabilities of one-loop anomalous dimension matrices corresponding to SU(2) sectors. This integrabilities hold for sectors corresponding to broken symmetries, as well as for a sector corresponding to a global symmetry. Then, we show agreements between string energies and anomalous dimensions for rational cases. In section 3, we generalize this analysis to SU(2) subsectors of more general orbifolds including $\mathcal{N}=0,1$ cases. Then, an agreement between sigma models of both sides is confirmed. In section 4, we extend this agreement of sigma models to SU(3) subsectors. In section 5, we conclude this paper.

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2. SU(2) sectors: $\mathcal{N}=2$ cases

We consider $\mathbb{C}\times\mathbb{C}^2/\mathbb{Z}_M$ orbifolds in this section. The coordinate of $\mathbb{C}^3$ is $(X,\ Y,\ Z)$. There is SO(6) rotation symmetry on $\mathbb{C}^3$. This group have subgroup $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R\times\mathrm{U}(1)_R$. $\mathrm{U}(1)_R$ is the rotation in the Z-plane. $(X,\ Y)$ forms a doublet of $\mathrm{SU}(2)_L$ and $(X,\ Y^\dagger)$ forms a doublet of $\mathrm{SU}(2)_R$. We take $\mathbb{Z}_M\subset\mathrm{SU}(2)_L$ and this choice breaks supersymmetry to $\mathcal{N}=2$. Our orbifold is described as $\mathbb{C}^3$ with the following identification,

$$(X,Y,Z)\sim(wX,w^{-1}Y,Z)\,,\qquad w=e^{2\pi i/M}\,. \tag{2.1}$$

The above orbifolding breaks the rotation symmetry to $\mathrm{U}(1)_L\times\mathrm{SU}(2)_R\times\mathrm{U}(1)_R$.

The near horizon geometry of the D3-branes located on the fixed point of the orbifold is $AdS_5\times S^5/\mathbb{Z}_M$[35]. The CFT dual of this background is a $\mathcal{N}=2$ quiver gauge theory.

2.1 Semiclassical string rotating in $S^5/\mathbb{Z}_M$

We consider semiclassical string configuration in $AdS_5\times S^5/\mathbb{Z}_M$. Especially, we are interested in configurations of twisted sectors. In fact, there are a simple generalization of semiclassical configuration in [7] to solutions in the twisted sectors. In [7], they treat the $O(6)\times\mathrm{SO}(4,2)$ sigma model as a bosonic part of strings propagating in $AdS_5\times S^5$. When considering classical motion of the strings which is nontrivial only in S^5, the action is reduced to time direction in AdS_5 and the $O(6)$ sigma model.

$$S=\frac{R^2}{2\pi\alpha'}\int d\tau d\sigma\left(\frac{1}{2}(\partial_a t)(\partial^a t)-\frac{1}{2}(\partial_a X_\mu)(\partial^a X_\mu)+\frac{1}{2}\Lambda(X_\mu X_\mu-1)\right), \tag{2.2}$$

where $\mu=1,\dots,6$ and

$$R^4=4\pi g_s NM\alpha'^2,\quad g_{QGT}^2=4\pi g_s M\,. \tag{2.3}$$

Hence, the overall factor of the action is $\frac{R^2}{2\pi\alpha'}=\frac{\sqrt{g_{QGT}^2 N}}{2\pi}\equiv\frac{\sqrt{\lambda}}{2\pi}$. Equations of motion are

$$\partial^2 t=0\,,\qquad \partial^2 X_\mu+\Lambda X_\mu=0\,,\qquad X_\mu X_\mu=1\,. \tag{2.4}$$

One can take the following ansatz,

$$\begin{aligned} t&=\kappa\tau, \qquad X\equiv X_1+iX_2=r_1(\sigma)e^{i(\alpha_1(\sigma)+w_1\tau)}\,,\\ Y\equiv X_3+iX_4&=r_2(\sigma)e^{i(\alpha_2(\sigma)+w_2\tau)}\,,\qquad Z\equiv X_5+iX_6=r_3(\sigma)e^{i(\alpha_3(\sigma)+w_3\tau)}\,. \end{aligned}\tag{2.5}$$

$r_i,\ \alpha_i$ satisfy the following periodicity conditions.

$$r_i(\sigma+2\pi)=r_i(\sigma),\quad \alpha_i(\sigma+2\pi)=\alpha_i(\sigma)+2\pi m_i\,, \tag{2.6}$$

where

$$m_1=\frac{k}{M}+\tilde{m}_1\,,\qquad m_2=-\frac{k}{M}+\tilde{m}_2\,,\qquad m_3=\tilde{m}_3\,,\qquad \tilde{m}_i\in Z,\ k=0,1,\dots,M-1\,. \tag{2.7}$$

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In S^5 case, m_i takes integer value, but in orbifold case we can take fractional values corresponding to twisted sectors. With this difference, remaining arguments is the same as that of S^5 case.

The most simple examples are circular strings. We consider these configurations now. These solutions take the following forms,

$$r_i(\sigma) = a_i = \text{const}\,, \qquad \alpha_i = m_i\sigma, \quad \Lambda = -\nu^2\,. \tag{2.8}$$

The equations of motion impose the relations between parameters,

$$\omega_i^2 = m_i^2 + \nu^2\,, \qquad \sum_{i=1}^{3} a_i^2 = 1\,. \tag{2.9}$$

Virasoro constraints become

$$\kappa^2 = \sum_{i=1}^{3} a_i^2(\omega_i^2 + m_i^2)\,, \qquad \sum_{i=1}^{3} a_i^2\omega_i m_i = 0\,. \tag{2.10}$$

A string energy E and angular momentums J_i are expressed in terms of sigma model variables,

$$\begin{aligned}\mathcal{E} &\equiv \frac{E}{\sqrt{\lambda}} = \int \frac{d\sigma}{2\pi}\dot{t} = \kappa\,,\\ \mathcal{J}_i &\equiv \frac{J_i}{\sqrt{\lambda}} = \int \frac{d\sigma}{2\pi}(X_{2i-1}\dot{X}_{2i} - X_{2i}\dot{X}_{2i-1}) = a_i^2\omega_i\,, \quad (i=1,2,3)\end{aligned} \tag{2.11}$$

J_1, J_2 and J_3 are the rotations in X, Y, Z-planes respectively. We define a total spin $L = \sum_{i=1}^{3}|J_i|$.

The relations (2.10) and (2.9) can be rewritten in terms of the energy and the spins (2.11),

$$\mathcal{E}^2 = 2\sum_{i=1}^{3}\sqrt{m_i^2+\nu^2}|\mathcal{J}_i| - \nu^2\,, \qquad \sum_{i=1}^{3}\frac{|\mathcal{J}_i|}{\sqrt{m_i^2+\nu^2}} = 1\,, \qquad \sum_{i=1}^{3} m_i\mathcal{J}_i = 0\,. \tag{2.12}$$

In order to describe $\mathcal{E}$ as a function of m_i and $\mathcal{J}_i$, firstly using the second equation we write ν as a function of m_i and $\mathcal{J}_i$ in large $\mathcal{J} \equiv L/\sqrt{\lambda}$ expansion. Then, inserting this ν to the first equation, we obtain $\mathcal{E}$ as a function of m_i, $\mathcal{J}_i$. The third equation is imposed at final stage.

In orbifold theory, we have to restrict string states to invariant states respect to orbifold action $e^{2\pi i(J_1-J_2)}$. So, a projection condition is

$$J_1 - J_2 = 0 \bmod M\,. \tag{2.13}$$

In order to compare string energies with anomalous dimensions of CFT operators in SU(2) subsector, we need 2-spin solutions of strings. Here, we firstly consider string configurations extending in X and Y directions and then consider configurations extending in X and Z directions.

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$\mathcal{J}_1, \mathcal{J}_2 \neq 0,\ \mathcal{J}_3 = 0, a_3 = 0$ **case.** The string energy has the following expansion in large $\mathcal{J}$,

$$\mathcal{E} = \mathcal{J} + \frac{|m_1||m_2|}{2\mathcal{J}} + \cdots . \tag{2.14}$$

The third equation of (2.12) becomes

$$m_1 \mathcal{J}_1 + m_2 \mathcal{J}_2 = 0 . \tag{2.15}$$

When $\mathcal{J}_1 > 0 > \mathcal{J}_2$, m_2 has the same sign as m_1. Therefore the parameters $\tilde{m}_i$ should satisfy the condition ($\tilde{m}_1 \geq 0,\ \tilde{m}_2 > 0$) or ($\tilde{m}_1 < 0,\ \tilde{m}_2 \leq 0$). The energy has the form,

$$E = L + \frac{\lambda}{2L}\left(\tilde{m}_1 + \frac{k}{M}\right)\left(\tilde{m}_2 - \frac{k}{M}\right) . \tag{2.16}$$

In field theory side, by an argument of classical scaling dimension and charges, the CFT operators dual to these strings should have J_1 Xs and $|J_2|$ $Y^\dagger$s. This sector correspond to $\mathrm{SU}(2)_R$ global symmetry of the orbifold theory. We will see spin chain corresponding to this sector in section 2.2.2.

When $\mathcal{J}_1, \mathcal{J}_2 > 0$, then m_2 has the opposite sign to m_1. In this case, ($\tilde{m}_1 \geq 0,\ \tilde{m}_2 \leq 0$) or ($\tilde{m}_1 < 0,\ \tilde{m}_2 > 0$) should be satisfied. The string energy is

$$E = L + \frac{\lambda}{2L}\left(\tilde{m}_1 + \frac{k}{M}\right)\left(\hat{m}_2 + \frac{k}{M}\right) , \tag{2.17}$$

where $\hat{m}_2 \equiv -\tilde{m}_2$, so that $\hat{m}_2$ has the same sign as $\tilde{m}_1$. By an argument of classical dimension and charges, these string configurations should correspond to $\mathrm{SU}(2)_L$ sector consisting of fields X and Y of the field theory. The $\mathrm{SU}(2)_L$ global symmetry of the original theory is broken in the orbifold theory. We will see spin chain of this sector in section 2.2.3.

$\mathcal{J}_1, \mathcal{J}_3 \neq 0,\ \mathcal{J}_2 = 0,\ a_2 = 0$ **case.** With replacing m_2, $\mathcal{J}_2$ by m_3, $\mathcal{J}_3$ respectively, the equations (2.14), (2.15) hold. We assume $J_1, J_3 > 0$. In this case, $m_1 = \tilde{m}_1 + \frac{k}{M}$, $m_3 = \tilde{m}_3$, $\tilde{m}_i \in Z$, and ($\tilde{m}_1 \geq 0,\ \tilde{m}_3 < 0$) or ($\tilde{m}_1 < 0,\ \tilde{m}_3 > 0$) should be satisfied. The energy becomes

$$E = L + \frac{\lambda}{2L}\left(\tilde{m}_1 + \frac{k}{M}\right)\hat{m}_3 , \tag{2.18}$$

where $\hat{m}_3 \equiv -\tilde{m}_3$, so that $\hat{m}_3$ has the same sign as $\tilde{m}_1$. These strings should correspond to the operators consisting of X and Z. We will see spin chain of this sector in the last paragraph of section 2.2.3.

2.2 Gauge theory side

2.2.1 General aspects

In this section, we discuss general aspects of the gauge side, not restricted to SU(2) sectors. We consider orbifold field theories originating from $\mathcal{N} = 4$ super Yang-Mills theory. Starting from $\mathrm{U}(NM)$ $\mathcal{N} = 4$ SYM, we project fields in $\mathcal{N} = 4$ theory onto Z_M invariant components. Fields surviving the projection satisfy

$$\phi = \gamma^\dagger (r \cdot \phi) \gamma \,, \tag{2.19}$$

where $\gamma = \mathrm{diag}(1, \omega, \ldots, \omega^{M-1})$ acting on gauge index and r is representation of orbifold action on fields with internal space indices. At the beginning, all fields in $\mathcal{N} = 4$ SYM are adjoint representation and $NM \times NM$ matrices. Due to this projection, the gauge groups become $\mathrm{U}(N)_1 \times \mathrm{U}(N)_2 \times \cdots \times \mathrm{U}(N)_M$. The Z_M actions on the $\mathbb{C}^3$ for $\mathcal{N} = 2$ case are indicated in (2.1). So, the projection conditions for the complex scalar fields are

$$\gamma^\dagger X \gamma = \omega X \,, \qquad \gamma^\dagger Y \gamma = \omega^{-1} Y \,, \qquad \gamma^\dagger Z \gamma = Z \,. \tag{2.20}$$

The components which survive the projection are the following.

$$Z = \begin{pmatrix} Z_{11} & & & \\ & Z_{22} & & \\ & & \ddots & \\ & & & Z_{MM} \end{pmatrix}, X = \begin{pmatrix} 0 & X_{12} & & \\ & 0 & X_{23} & \\ & & \ddots & \ddots \\ X_{M1} & & & \end{pmatrix}, Y = \begin{pmatrix} 0 & & & Y_{1M} \\ Y_{21} & 0 & & \\ & Y_{32} & 0 & \\ & & \ddots & \ddots \end{pmatrix}. \tag{2.21}$$

Each component is $N \times N$ matrix and is complex scalar part of chiral superfield in terms of $\mathcal{N} = 1$ supersymmetry. $Z_{i,i}$ is adjoint scalar field of the i-th U(N) gauge group and scalar component of vector multiplet in $\mathcal{N} = 2$ context. $X_{i,(i+1)}$ is bifundamental representation of gauge groups, N for i-th gauge group and $\bar{N}$ for $i+1$-th gauge group.[1] $Y_{(i+1),i}$ is bifundamental representation, $\bar{N}$ for i-th gauge group and N for $i+1$-th gauge group. The two chiral superfields, corresponding to $X_{i,(i+1)}$ and $Y_{(i+1),i}$, form hypermultiplet in bifundamental representation in $\mathcal{N} = 2$ context. $X_{i,(i+1)}$ and the Hermite conjugate of $Y_{(i+1),i}$ form a doublet of $\mathrm{SU}(2)_R$ symmetry. This R-symmetry is a part of SO(6) R-symmetry of the original theory and remains as a global symmetry in the orbifold field theory. On the other hand, X and Y have formed a doublet of $\mathrm{SU}(2)_L \subset \mathrm{SO}(6)$ in the $\mathcal{N} = 4$ theory. However, this $\mathrm{SU}(2)_L$ symmetry is broken by orbifolding and is not global symmetry of the orbifold theory.

Arguments of the rest of this subsection is applied not only to the $\mathcal{N} = 2$ cases but also to more general orbifold cases in section 3.

In our paper, We does not decompose the $NM \times NM$ matrix to $N \times N$ matrix. That is, we treat the single trace operators which have the following form,

$$\mathrm{tr} XXYZY^\dagger X \cdots \,, \tag{2.22}$$

where X, Y and Z are $NM \times NM$ matrices of (2.21) and the trace is gauge index trace. Using this operator, we need not to specify which gauge group each field belongs to. More carefully, in order to take account of all gauge invariant operators we have to introduce the 'twisted sectors' of the operators,

$$\mathrm{tr} \gamma_k XXYZY^\dagger X \cdots \,, \tag{2.23}$$

[1] We denote this type of bifundamental representation as $(i, \overline{i+1})$.

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where

$$\gamma_k = \mathrm{diag}(1, w^k, w^{2k}, \ldots, w^{(M-1)k})\,, \qquad w = e^{2\pi i/M}\,, \qquad k = 1, 2, \ldots, M-1\,. \tag{2.24}$$

We call these operators twisted operators. On the other hand, the operators of (2.22) correspond to $k = 0$ and we call these untwisted operators.

The operators with different twist number k are not mixed in 2-point function. This is due to $\mathbb{Z}_M$ symmetry which orbifold field theory has. An element $g = e^{2\pi i/M}$ in this $\mathbb{Z}_M$ acts on arbitrary field $\phi_{i,j}$ of field theory as $g : \phi_{i,j} \to \phi_{i+1,j+1}$.($i$ and j denote i-th gauge group and j-th gauge group, respectively.) Then, this acts on single trace operator as $g : \mathrm{tr}\gamma_k XXY \cdots \to \omega^{-k}\mathrm{tr}\gamma_k XXY \cdots$. This means that operators with twist number k are eigenstate of this $\mathbb{Z}_M$ symmetry and have eigenvalue ω^{-k} for the element g. Then, 2-point function between operators with different twist number, vanishes. Hence, we can treat each sector separately.

The trace of operators requires that SO(6) charge $(J_1,\ J_2,\ J_3)$ of operators should satisfy $J_1 - J_2 = 0 \mod M$ in the $\mathcal{N} = 2$ case. If otherwise, diagonal parts of the operators are zero matrices and the operators vanish by the trace. This fact can be seen as following.

$$\begin{aligned}
\mathrm{tr}\gamma_k XYXY^\dagger Z\cdots &= \mathrm{tr}\gamma_k\gamma_1^{-1}\gamma_1 XYXY^\dagger Z\cdots = \omega^{-1}\mathrm{tr}\gamma_k\gamma_1^{-1}X\gamma_1 YXY^\dagger Z\cdots \\
&= \omega^{-(1-1)}\mathrm{tr}\gamma_k\gamma_1^{-1}XY\gamma_1 XY^\dagger Z\cdots = \cdots \\
&= \omega^{-(J_1-J_2)}\mathrm{tr}\gamma_1\gamma_k\gamma_1^{-1}XYXY^\dagger Z\cdots \\
&= \omega^{-(J_1-J_2)}\mathrm{tr}\gamma_k XYXY^\dagger Z\cdots\,.
\end{aligned} \tag{2.25}$$

So, the invariant condition (2.13) is automatically satisfied in field theory side. It is easy to generalize this fact to more general $\mathbb{C}^3/\mathbb{Z}_M$ orbifold cases in section 3.

In the planar limit, anomalous dimension matrix of untwisted operators ($k = 0$) in orbifold theory is the same as that of $\mathcal{N} = 4$ theory. In orbifold theory, vertices of Feynman diagram are inherited from $\mathcal{N} = 4$ SYM. Hence, the number of fields of each Lorentz type which run loops are the same as that of diagrams in $\mathcal{N} = 4$ U(N) SYM. Then, we obtain the same anomalous dimension matrix as $\mathcal{N} = 4$ SU(N) SYM case. We explain this fact in Appendix. This agreement could also be explained by 'inherited principle' of [37, 36].

However, for twisted sectors of operators ($k \neq 0$) there is a subtlety about interactions between the L-th site and the first ($(L+1)$-th) site. That is, the interactions involving the L-th site and first site step the matrix γ_k and may generate additional phase shift. Hence, the Feynman diagrams which have an interaction between the L-th site and the first site may have these phase shifts. Whether this shift occurs or not depends on an order of gauge group at the interaction point. When the phase shifts occur in some of these diagrams, the anomalous dimension matrix is not the same as that of untwisted sector. Nevertheless, we can resolve this subtlety and we find that the spin chain systems corresponding to these twisted sectors of operators are also solvable. We will see concrete examples of this fact in the following subsections.

2.2.2 XXX spin chain corresponding to $\mathrm{SU}(2)_R$ symmetry

Let us consider SU(2) subsectors. SU(2) subsectors, not only $\mathrm{SU}(2)_R$ but also others, are closed for loop corrections, the feature inherited from the $\mathcal{N}=4$ SYM [22]. We consider operators of $\mathrm{SU}(2)_R$ subsector in this subsection. This $\mathrm{SU}(2)_R$ is a global symmetry of the orbifold theory we are considering. The single trace operators in $\mathrm{SU}(2)_R$ subsector have the following form

$$\mathrm{Tr}\gamma_k X^{J_1} Y^{\dagger |J_2|} + \text{permutations}\,, \qquad (J_1 > 0,\ J_2 < 0) \tag{2.26}$$

X and $Y^\dagger$ form a doublet of $\mathrm{SU}(2)_R$. These operators have the SO(6) charges $(J_1,\ J_2,\ 0)$ and should correspond to the strings with these charges. The circular strings with these charges are constructed in the above and have the energy (2.16).

We treat X as an up-spin and $Y^\dagger$ as a down-spin. More precisely, for the first site, $\uparrow_{\text{1st site}}(\downarrow_{\text{1st site}})$ is identified as $\gamma_k X(\gamma_k Y^\dagger)$. Periodic boundary condition $\uparrow_{\text{L+1th site}}=\uparrow_{\text{1st site}}$, $\downarrow_{\text{L+1th site}}=\downarrow_{\text{1st site}}$ is imposed. We denote $\mathrm{tr}(\gamma_k XY^\dagger XXY^\dagger\cdots)$ as $|\uparrow\downarrow\uparrow\uparrow\downarrow\cdots\rangle_k$, where a suffix k means that this state corresponds to a operator with twist number k. We assume $J_1 > |J_2| > 0$ and take $|\uparrow\uparrow\cdots\uparrow\rangle_k$ as ground state. Both of $X,\ Y^\dagger$ are $(i,\overline{i+1})$ type bi-fundamentals. So, the sequence of gauge groups i, which denote i-th gauge group, in the trace is one-way. The phase shift for twisted sector mentioned above does not happen in this case (See figure 1). Then a one-loop anomalous dimension matrix for any twist number sector of operators is the same as that of SU(2) subsector in $\mathcal{N}=4$ SYM.

$$D_{\text{1-loop}} = \frac{\lambda}{16\pi^2}\sum_{l=1}^{L}(1-\vec{\sigma}_l\cdot\vec{\sigma}_{l+1})\,, \tag{2.27}$$

where $\vec{\sigma}_l$ are the Pauli matrices which act on the spin at the l-th site. This takes the form of a hamiltonian of the $\mathrm{XXX}_{1/2}$ Heisenberg spin chain. The boundary condition is periodic now. This model is solvable by Bethe ansatz. Bethe equations are

$$\left(\frac{u_i+i/2}{u_i-i/2}\right)^L = \prod_{j=1,j\neq i}^{|J_2|}\frac{u_i-u_j+i}{u_i-u_j-i}\,. \tag{2.28}$$

Compared to $\mathcal{N}=4$ case, Only difference is in a translation condition. In the presence of γ_k, the cyclicity of the trace imposes the following condition,

$$\mathrm{tr}\gamma_k\phi_1\cdots\phi_L = \mathrm{tr}\phi_L\gamma_k\phi_1\cdots = w^k\mathrm{tr}\gamma_k\phi_L\phi_1\cdots\,, \tag{2.29}$$

where ϕ_i represent X or $Y^\dagger$ field. So, the translation condition for the spin chain states becomes

$$T|\text{state}\rangle_k = w^{-k}|\text{state}\rangle_k\,, \tag{2.30}$$

where $T=e^{iP}$ is the operator which translate each site to the right by a site. In terms of Bethe roots this condition is described as

$$e^{iP} = \prod_{j=1}^{|J_2|}\frac{u_j+i/2}{u_j-i/2} = w^{-k}\,. \tag{2.31}$$

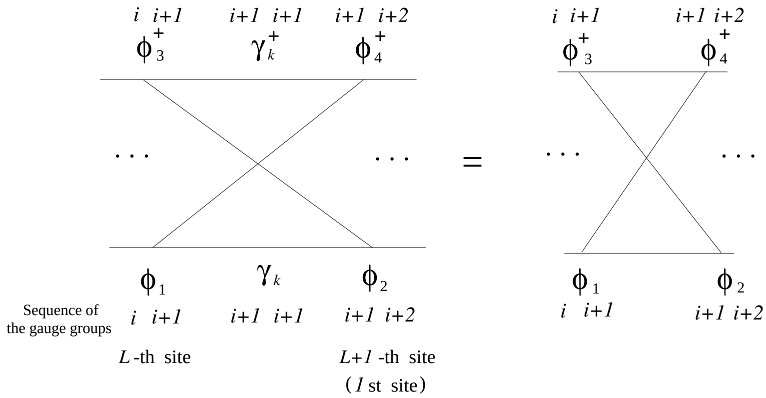

Figure 1: Interaction between the L-th site and the first ($(L+1)$-th) site. Each ϕ is X or $Y^\dagger$ and i, $i+1$ and $i+2$ represent the gauge groups.

The anomalous dimensions, which are proportional to energies of the spin chain, take the following form,

$$\gamma = \frac{\lambda}{8\pi^2}\sum_{i=1}^{|J_2|}\frac{1}{u_i^2+1/4}\,. \tag{2.32}$$

Rescaling $u_i = Lx_i$ and taking large L limit, the logarithms of (2.28), (2.31) become

$$\frac{1}{x_i} = \frac{2}{L}\sum_{j=1,j\neq i}^{|J_2|}\frac{1}{x_i - x_j} - 2\pi n\,, \qquad n\in\mathbb{Z} \tag{2.33}$$

$$\frac{1}{L}\sum_{i=1}^{|J_2|}\frac{1}{x_i} = -2\pi\left(m+\frac{k}{M}\right), \qquad m\in\mathbb{Z}\,, \tag{2.34}$$

where we assume that all mode numbers of the Bethe equations are n. The anomalous dimension of (2.32) becomes in the large L expansion,

$$\gamma = \frac{\lambda}{8\pi^2 L^2}\sum_{i=1}^{|J_2|}\frac{1}{x_i^2}\,. \tag{2.35}$$

The equations (2.33), (2.34) can be solved by various methods[21, 43]. The solutions are rational type. We will outline one of the methods in Appendix. Using these solutions, we obtain the anomalous dimensions from (2.35).(Also see Appendix.) Leading terms of these in large L expansion are

$$\gamma = \frac{\lambda}{2L}\left(m+\frac{k}{M}\right)\left(n-m-\frac{k}{M}\right). \tag{2.36}$$

Note that $(m+k/M)$ has the same sign as n and $(n-(m+k/M))$ also has the same sign as n.(See Appendix.) Hence, identifying that $m_1 = m+k/M, m_2 = n-(m+k/M)$ and remembering that m_2 has the same sign as m_1, this result agrees with (2.16). Especially, we can see from this result that the twisted sectors of the strings are dual to the twisted sectors of the operators.

2.2.3 XXX spin chain with twisted boundary condition corresponding to broken SU(2) symmetry

Next, we consider spin chains corresponding to SU(2) symmetries broken by orbifolding. One of examples is the $\mathrm{SU}(2)_L$ sector, which consists of the following operators,

$$\mathrm{tr}\gamma_k X^{J_1} Y^{J_2} + \text{permutations}\,, \qquad (J_1,\ J_2 > 0) \tag{2.37}$$

X and Y fields form a doublet of $\mathrm{SU}(2)_L$ symmetry, which is broken by orbifolding. These operators should correspond to string states with charges $J_1,\ J_2 > 0$. Such a string state was constructed in the previous section and the string energy was (2.17). We identify X(Y) as up-spin(down-spin). As like in $\mathrm{SU}(2)_R$ case, the first site state $\uparrow_{\text{1st site}}(\downarrow_{\text{1st site}})$ is identified $\gamma_k X(\gamma_k Y)$. A periodic boundary condition between the first site and the $L+1$th site is imposed. We assume $J_1 > J_2$ and take $|\uparrow\uparrow\cdots\uparrow\rangle_k$ as the ground state.

X field is a $(i,\ \overline{i+1})$-type operator but Y field is a $(i,\ \overline{i-1})$-type operator. Hence the sequence of the gauge groups in the trace of the operator is not one-way but zigzag. This causes the phase shifts of interactions between the L-th site and the first ($(L+1)$-th) site. Indeed the interactions which interchange an up-spin and a down-spin generate the phase shifts. At these interactions, the sequence of the gauge groups changes between two traces of two operators which correlate with each other. This difference generates phase shift. These type of interaction vertices are included by F-term potential $\mathrm{tr}|[X,Y]|^2$. The interactions which generate the phase shifts are shown in figure 2. On the other hand if the interaction is proportional to the identity operator on the spin chain states, the sequence of the gauge groups does not change between two traces and the phase shift does not occur.

In the existence of the phase shifts, the one-loop anomalous dimension matrix does not agree with XXX-spin chain hamiltonian (2.27). However, by changing the basis of the spin chain to

$$\begin{aligned} |\tilde{\downarrow}_{\text{L+1-th site}}\rangle_k &= u w^{-2k} |\downarrow_{\text{L+1-th site}}\rangle_k\,, \qquad |\tilde{\uparrow}_{\text{L+1-th site}}\rangle_k = u |\uparrow_{\text{L+1-th site}}\rangle_k\,, \\ |\tilde{\downarrow}_{\text{i-th site}}\rangle_k &= |\downarrow_{\text{i-th site}}\rangle_k\,, \qquad |\tilde{\uparrow}_{\text{i-th site}}\rangle_k = |\uparrow_{\text{i-th site}}\rangle_k, \quad (i = 1, \ldots, L) \end{aligned} \tag{2.38}$$

the phase shifts are cancelled and the one-loop anomalous dimension matrix becomes the same as (2.27). u is an arbitrary phase. In other words, we can regard our model as XXX spin chain of $|\tilde{\uparrow}\rangle$ and $|\tilde{\downarrow}\rangle$ with hamiltonian (2.27) and with twisted boundary condition,

$$|\tilde{\downarrow}_{\text{1st site}}\rangle_k = u^{-1} w^{2k} |\tilde{\downarrow}_{\text{L+1-th site}}\rangle_k\,, \qquad |\tilde{\uparrow}_{\text{1st site}}\rangle_k = u^{-1} |\tilde{\uparrow}_{\text{L+1-th site}}\rangle_k\,. \tag{2.39}$$

The hamiltonian preserves SU(2) symmetry, but this boundary condition breaks this symmetry. This system is exactly solvable by Beth ansatz with a slight modification [30]. Since it is convenient for the ground state to be periodic along the spin chain, we choose $u = 1$. With the twisted boundary condition, Bethe equations become

$$w^{2k} \left(\frac{u_i + i/2}{u_i - i/2} \right)^L = \prod_{j=1, j\neq i}^{J_2} \frac{u_i - u_j + i}{u_i - u_j - i}\,. \tag{2.40}$$

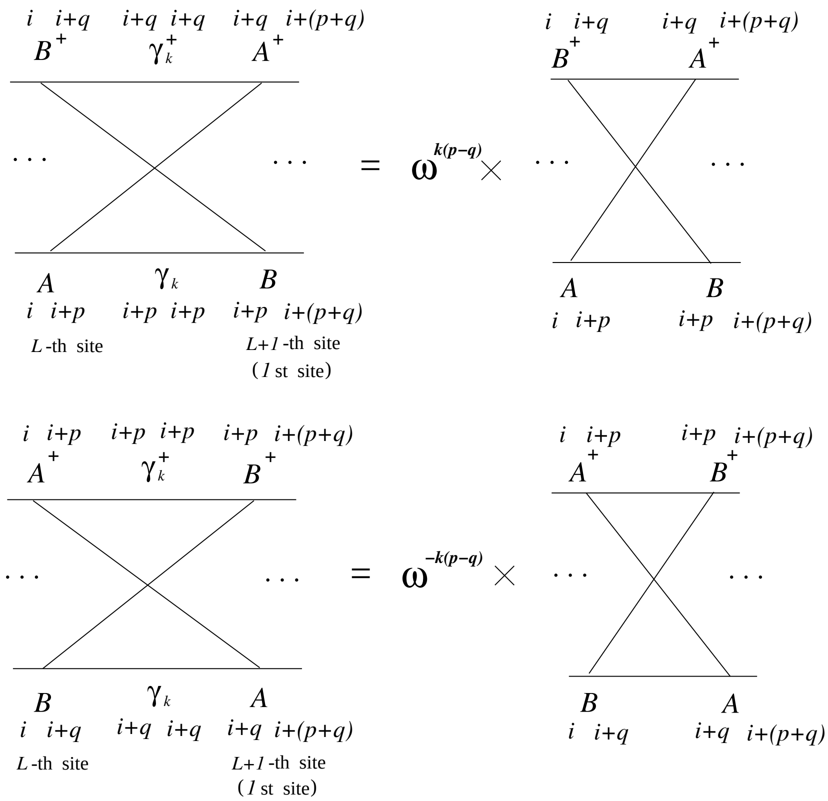

Figure 2: The interactions which generate the phase shift at the end (the beginning) of the chain. A and B are fields consisting of operators. A is $(i, \overline{i+p})$-type and B is $(i, \overline{i+q})$-type operator. In the $\mathrm{SU}(2)_L$ case, $A = X,\ B = Y$ and $p = 1, q = -1$.

Only difference to the periodic case is phase factor w^{2k} in the l.h.s., which reflects twisted boundary condition (2.39). The cyclicity of trace indicates

$$\mathrm{tr}\gamma_k XY \cdots \phi = w^{k\sigma_3^\phi} \mathrm{tr}\gamma_k \phi XY \cdots , \tag{2.41}$$

where ϕ is X or Y field and $\sigma_3^X = 1$ and $\sigma_3^Y = -1$. This relation can be rewritten in terms of the spin chain,

$$\begin{aligned} |\uparrow\downarrow \cdots \phi\rangle_k &= w^{k\sigma_3^\phi} |\phi \uparrow\downarrow \cdots\rangle_k \\ &= w^{k\sigma_3^\phi} w^{k(1-\sigma_3^\phi)} T |\uparrow\downarrow \cdots \phi\rangle_k = w^k T |\uparrow\downarrow \cdots \phi\rangle_k \,, \end{aligned} \tag{2.42}$$

where ϕ represents $\uparrow$ or $\downarrow$ appropriately. When going to the second line, we used the twisted boundary condition, $|\phi_{\text{1st site}}\rangle_k = w^{k(1-\sigma_3^\phi)} |\phi_{\text{L+1-th site}}\rangle_k$. Using the Bethe roots, the equation (2.42) can be written as

$$e^{iP} = \prod_{j=1}^{J_2} \frac{u_j + i/2}{u_j - i/2} = w^{-k} \,. \tag{2.43}$$

An expression of the energy in terms of the Bethe roots is the same as (2.32). Taking logarithms of (2.40) and (2.43) and taking large L expansion, we obtain

$$\frac{1}{x_i} = \frac{2}{L}\sum_{j=1,j\neq i}^{J_2}\frac{1}{x_i - x_j} - 2\pi\left(n + 2\frac{k}{M}\right) \qquad n \in \mathbb{Z} \tag{2.44}$$

$$\frac{1}{L}\sum_{i=1}^{J_2}\frac{1}{x_i} = -2\pi\left(m + \frac{k}{M}\right), \qquad m \in \mathbb{Z}\,, \tag{2.45}$$

where we rescale $u_i = Lx_i$. Solving these equations and using (2.35), we obtain leading terms of the anomalous dimensions in the large L expansion,

$$\gamma = \frac{\lambda}{2L}\left(m + \frac{k}{M}\right)\left(n - m + \frac{k}{M}\right). \tag{2.46}$$

Note that $m + k/M$, $n + 2k/M$ and $(n + 2k/M) - (m + k/M)$ all have the same sign.(See Appendix.) So, identifying $m_1 = (m + \frac{k}{M})$, $-m_2 = (n + 2\frac{k}{M}) - (m + \frac{k}{M})$ and remembering m_2 has the opposite sign to m_1, the result (2.46) agrees with (2.17). Also in this case, we can see that the twisted string states correspond to the twisted sectors of the operators.

X, Z case. Let us look at one more example of spin chains associated with other broken SU(2)s. We consider the operators consisting of X and Z fields, which form a doublet of $SU(2) \subset SO(6)$ in the original $\mathcal{N} = 4$ theory. This SU(2) is broken in the orbifold theory we are considering. The argument is almost parallel to that of $SU(2)_L$ case. The spin chain corresponding to this (X, Z) system is a XXX spin chain of the hamiltonian (2.27) with twisted boundary condition, which has the following properties.[2]

$$T|\text{state}\rangle_k = w^{-k}|\text{state}\rangle_k\,, \tag{2.47}$$

$$|\uparrow_{\text{1st site}}\rangle_k = |\uparrow_{\text{(L+1)th site}}\rangle_k\,, \tag{2.48}$$

$$|\downarrow_{\text{1st site}}\rangle_k = w^k|\downarrow_{\text{(L+1)th site}}\rangle_k\,. \tag{2.49}$$

The anomalous dimensions can be obtained by solving the corresponding Bethe ansatz and the result is

$$\gamma = \frac{\lambda}{2L}\left(m + \frac{k}{M}\right)(n - m)\,, \qquad m,\ n \in \mathbb{Z}\,. \tag{2.50}$$

Here, $m + k/M$, $n + k/M$ and $(n + k/M) - (m + k/M)$ have the same sign. Then, identifying $m_1 = m + k/M$ and $-m_3 = (n + k/M) - (m + k/M)$ and remembering that m_3 has the opposite sign to m_1, this agrees with (2.18).

3. SU(2) sectors of more general $\mathbb{C}^3/Z_M$ orbifolds.

In this section, we show that the integrabilities of the gauge side and the agreements between the string side and the gauge side in SU(2) sectors hold even in $\mathcal{N} = 1$ or non-supersymmetric orbifolds. Hence, we allow $\mathbb{Z}_M$ orbifold action arbitrary. That is, we consider orbifold $(X,\ Y,\ Z) \sim (\omega^{p_X}X,\ \omega^{p_Y}Y,\ \omega^{p_Z}Z)$, $(p_X, p_Y, p_Z \in \mathbb{Z})$.

[2] From here on, even when we change a basis of spin chain to a new set, we denote new basis as $\uparrow$, $\downarrow$ without tilde.

3.1 Correspondence of rational solutions

We consider a SU(2) subsector consisting of complex scalars $A,\ B$. Let the identification by orbifolding on these fields be $(A,B) \sim (\omega^p A, \omega^q B)$. Here, the integers $p,\ q$ take values in $[-(M-1),(M-1)]$. In the $\mathcal{N}=2$ cases of the previous section, the $\mathrm{SU}(2)_R$ case corresponds to $A=X,\ B=Y^\dagger$ and $p=q=1$, the $\mathrm{SU}(2)_L$ case to $A=X,\ B=Y$ and $p=1,\ q=-1$ and the (X,Z) case to $A=X,\ B=Z$ and $p=1,\ q=0$.

First, let us consider the string side. We consider 2-spin circular strings as in the case of $\mathcal{N}=2$. The generalization of the result of the previous section is straightforward. The string energy in present case becomes

$$\begin{aligned} E-L &= -\frac{\lambda}{2L} m_A m_B\,, \\ m_A &= \left(\tilde{m}_A + \frac{p}{M}k\right), \qquad -m_B = \left(\tilde{m}_B - \frac{q}{M}k\right), \qquad \tilde{m}_A, \tilde{m}_B \in \mathbb{Z}\,. \end{aligned} \tag{3.1}$$

Here we assume $J_A, J_B > 0$. Hence, m_B needs to have the opposite sign to m_A.

Next let us go to the field theory side. A is $(i,\overline{i+p})$-type bifundamental and B is $(i,\overline{i+q})$-type bifundamental of the gauge groups. We regard $A(B)$ as up-spin(down-spin). We assume $J_A > J_B$ and take $|\uparrow\uparrow\cdots\uparrow\rangle_k$ as ground state. In this theory, supersymmetry may not exist. However, vertices of interactions are inherited from the original theory and there are two types of scalar potentials, one from D-term and the other from F-term. Like in the $\mathcal{N}=2$ case, the vertices from the D-term potential are proportional to identity matrix in the spin chain system. The F-term includes the vertices which permute neighbouring spins. These interactions generate the phase shifts between the L-th site and the first site, if $p-q \neq 0 \mod M$.(See figure 2.) These phase shifts can be cancelled by the twisted boundary condition as in the previous case,

$$|\downarrow_{\text{1st site}}\rangle_k = w^{k(p-q)}|\downarrow_{\text{L+1-th site}}\rangle\,, \qquad |\uparrow_{\text{1st site}}\rangle = |\uparrow_{\text{L+1-th site}}\rangle\,. \tag{3.2}$$

The hamiltonian of this spin chain is (2.27). The cyclicity of the trace means

$$\begin{aligned} \mathrm{tr}\gamma_k \cdots A &= w^{kp}\mathrm{tr}\gamma_k A \cdots\,, \\ \mathrm{tr}\gamma_k \cdots B &= w^{kq}\mathrm{tr}\gamma_k B \cdots\,. \end{aligned} \tag{3.3}$$

So, a translation condition becomes

$$T|\text{state}\rangle_k = w^{-kp}|\text{state}\rangle_k\,. \tag{3.4}$$

From these settings, we can obtain Bethe equations and a translation condition in terms of Bethe roots,

$$\omega^{k(p-q)}\left(\frac{u_i + i/2}{u_i - i/2}\right)^L = \prod_{j=1, j\neq i}^{J_B} \frac{u_i - u_j + i}{u_i - u_j - i}\,, \tag{3.5}$$

$$\prod_{j=1}^{J_B} \frac{u_j + i/2}{u_j - i/2} = \omega^{-kp}\,. \tag{3.6}$$

Taking logarithms of these equations and taking large L limit, these equations become

$$\frac{1}{x_i} = \frac{2}{L}\sum_{j=1,j\neq i}^{J_B}\frac{1}{x_i - x_j} - 2\pi\left(n + k\frac{p-q}{M}\right), \qquad n \in \mathbb{Z}$$

$$\frac{1}{L}\sum_{i=1}^{J_B}\frac{1}{x_i} = -2\pi\left(m + k\frac{p}{M}\right), \qquad m \in \mathbb{Z}\,. \tag{3.7}$$

Here we rescale $u_i = Lx_i$. The expression of anomalous dimensions in terms of Bethe roots is the same as in the previous examples. Solving above equations, the anomalous dimension becomes

$$\gamma = \frac{\lambda}{2L}\left(m + \frac{p}{M}k\right)\left(n - m - \frac{q}{M}k\right), \qquad m,\ n \in \mathbb{Z}\,. \tag{3.8}$$

Notice that $(m + \frac{p}{M}k)$, $(n + \frac{p-q}{M}k)$ and $(n + \frac{p-q}{M}k) - (m + \frac{p}{M}k)$ have the same sign.(See Appendix.) Hence, identifying $m_A = m + \frac{p}{M}k$, $-m_B = (n + \frac{p-q}{M}k) - (m + \frac{p}{M}k)$ and remembering that m_B has the opposite sign to m_A, this agrees with the result of the string side (3.1).

3.2 Agreement of sigma models from string side and gauge side

In this subsection, we will see an agreement of sigma models of both sides. From the spin chain side, we can obtain a sigma model action by a continuum limit. On the string side, an action which describe a string with large angular momentum in a S^3 part of S^5 is obtained. These two actions are the same as each other in the $\mathcal{N} = 4$ case [15]. We generalize this to the orbifold cases. The argument is almost the same as that of the original case except for boundary conditions. Hence, we will not write details but things that are peculiar to orbifold cases.

The identification by orbifolding is $(A,\ B) \sim (\omega^p A,\ \omega^q B)$. The boundary condition for the spin chain is

$$|\uparrow_{\text{1st site}}\rangle_k = w^{-k(p-q)/2}|\uparrow_{\text{L+1-th site}}\rangle_k\,, \qquad |\downarrow_{\text{1st site}}\rangle_k = w^{k(p-q)/2}|\downarrow_{\text{L+1-th site}}\rangle_k\,, \tag{3.9}$$

where we shift an overall phase $e^{-k(p-q)/2}$ compared to (3.2) by using an arbitrary overall phase factor(See (2.39)). This is done in order to make boundary condition for $\phi(\sigma)$(see below) become well-defined. By the effect of the change of the overall phase, the translation condition is also changed to

$$T|\text{state}\rangle_k = e^{-k(p+q)/2}|\text{state}\rangle_k\,. \tag{3.10}$$

One can introduce a continuous set of states

$$|\vec{n}\rangle_k = e^{i\phi/2}\cos\frac{\theta}{2}|\uparrow\rangle_k - e^{-i\phi/2}\sin\frac{\theta}{2}|\downarrow\rangle_k\,. \tag{3.11}$$

Here, $\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Taking large number of sites limit and labelling positions of the sites by σ, functions $\phi(\sigma)$, $\theta(\sigma)$ become continuous. When going around the chain, a state $|\vec{n}\rangle$ should go back an initial state. Taking account of (3.9), this means that a boundary conditions for $\phi(\sigma)$ are

$$\phi(2\pi+\sigma)=\phi(\sigma)-2\pi\frac{k(p-q)}{M}\,. \tag{3.12}$$

The following sigma model action is obtained by path-integral[15],[3]

$$S=-\frac{L}{4\pi}\int d\sigma dt\cos\theta\dot{\phi}-\frac{\lambda}{16\pi L}\int d\sigma dt((\partial_\sigma\theta)^2+\sin^2\theta(\partial_\sigma\phi)^2\,. \tag{3.13}$$

The momentum condition (3.10) dictates that the momentum along the σ-direction becomes

$$P\equiv\frac{2\pi}{L}\int T_{01}d\sigma=-\frac{1}{2}\int\cos\theta\partial_\sigma\phi=-2\pi\frac{k(p+q)}{2M}\,. \tag{3.14}$$

Next, let us move to the string side. We are interested in $R_t\times S^3$ motion of the string. R_t is time direction of global coordinate of AdS_5 and S^3 is a part of S^5. The metric is

$$ds^2=-dt^2+d\Omega_3^2=-dt^2+d\psi^2+\cos\psi d\phi_1^2+\sin\psi d\phi_2^2\,. \tag{3.15}$$

$A,\ B$ are expressed in these coordinates as $A=\cos\psi e^{i\phi_1},\ B=\sin\psi e^{i\phi_2}$. The identification by orbifolding is $(\phi_1,\ \phi_2)\sim(\phi_1+2\pi\frac{p}{M},\ \phi_2+2\pi\frac{q}{M})$. One changes the coordinate to $\phi_1=\varphi_1+\varphi_2,\ \phi_2=\varphi_1-\varphi_2$, and obtains a metric

$$ds^2=-dt^2+d\psi^2+d\varphi_1^2+d\varphi_2^2+2\cos(2\psi)d\varphi_1d\varphi_2\,. \tag{3.16}$$

We are considering string motion with large angular momentum $L=J_1+J_2$. In the new coordinate, this corresponds to fast motion along φ_1. One can subtract motion of φ_1 direction from the Polyakov action in (3.16). A resulting sigma model action has the same form as the action (3.13) from the spin chain, if we identify $\varphi_2=-\frac{1}{2}\phi,\ \psi=\frac{1}{2}\theta$ [15]. Boundary conditions for $\varphi_1,\ \varphi_2$ are

$$\varphi_1(\sigma+2\pi)\ =\ \varphi_1(\sigma)+2\pi\frac{k(p+q)}{2M}\,, \tag{3.17}$$

$$\varphi_2(\sigma+2\pi)\ =\ \varphi_2(\sigma)+2\pi\frac{k(p-q)}{2M}\,. \tag{3.18}$$

The Virasoro constraint imposes

$$\int d\sigma\cos(2\psi)\partial_\sigma\varphi_2=-\int d\sigma\partial_\sigma\varphi_1=-2\pi\frac{k(p+q)}{2M}\,. \tag{3.19}$$

where we use the equation (3.17). Remembering $\varphi_2=-\frac{1}{2}\phi$, $\psi=\frac{1}{2}\theta$, the equations (3.18) and (3.19) agree with the equations (3.12) and (3.14), respectively. We confirmed the agreement between sigma models of both sides.

[3] We take $0\leq\sigma\leq 2\pi$. In [15], σ is taken from 0 to L.

4. SU(3) sectors

In this section, we investigate SU(3) subsectors of SO(6). We consider a sector of three complex scalars(coordinates) A, B and C, and take orbifold action so that identification by orbifolding becomes $(A,\ B,\ C) \sim (e^{2\pi i \frac{p}{M}} A,\ e^{2\pi i \frac{q}{M}} B,\ e^{2\pi i \frac{r}{M}} C)$. In the gauge theory side, we consider single trace operators of the following form,

$$\mathrm{tr}\gamma_k A^{J_1} B^{J_2} C^{J_3} + \text{permutations}. \tag{4.1}$$

A is in $(i,\ \overline{i+p})$ bi-fundamental representation of the gauge groups. B is a $(i,\ \overline{i+q})$-type and C is a $(i,\ \overline{i+r})$-type.

The argument of the one-loop anomalous dimension matrix is almost parallel to SU(2) cases. As explained in the previous section, the one-loop anomalous dimension matrix of the untwisted sector of the operators is the same as that of $\mathcal{N} = 4$ SYM, that is, a hamiltonian of the SU(3) XXX spin chain,

$$D_{\text{1-loop}} = \frac{\lambda}{8\pi^2} \sum_{l=1}^{l=L} (1 - P_{l,l+1}), \tag{4.2}$$

where $P_{l,l+1}$ is a permutation operator between the l-th site and the $l+1$-th site. A boundary condition is periodic. This model is solvable. On the other hand, the anomalous dimension matrices of the twisted sectors of the operators may have the phase shifts between the $L+1$-th site and the first site. As in the SU(2) cases, these phases can be cancelled by twisted boundary conditions. As a result, the twisted sectors are regarded as the SU(3) XXX spin chains (4.2) with the twisted boundary conditions.

4.1 Agreement of sigma models from both sides

Now, we extend the argument of the sigma model agreement in previous section to the SU(3) case. In the $\mathcal{N} = 4$ case, the SU(3) generalization was done in [18].

We regard A, B and C as spin chain states $|1\rangle$, $|2\rangle$ and $|3\rangle$, respectively. By the same argument as the SU(2) cases, we take twisted boundary conditions,

$$\begin{aligned} |1_{\text{1st-site}}\rangle_k &= e^{-2\pi i \frac{k(p-q)}{2M}} |1_{\text{L+1th-site}}\rangle_k\,, \qquad |2_{\text{1st-site}}\rangle_k = e^{2\pi i \frac{k(p-q)}{2M}} |2_{\text{L+1th-site}}\rangle_k\,, \\ |3_{\text{1st-site}}\rangle_k &= e^{2\pi i \frac{k}{M}(\frac{p+q}{2} - r)} |3_{\text{L+1th-site}}\rangle_k\,, \end{aligned} \tag{4.3}$$

A hamiltonian of this system is (4.2). In this boundary condition, the cyclicity of the trace means

$$T|\text{state}\rangle_k = e^{-2\pi i \frac{k(p+q)}{2M}} |\text{state}\rangle_k\,. \tag{4.4}$$

We introduce coherent states

$$|\vec{n}\rangle_k = \cos\frac{\theta_2}{2} \cos\frac{\theta_1}{2} e^{i\phi/2} |1\rangle_k - \cos\frac{\theta_2}{2} \sin\frac{\theta_1}{2} e^{-i\phi/2} |2\rangle_k + \sin\frac{\theta_2}{2} e^{i\chi/2} |3\rangle_k \tag{4.5}$$

In large number of sites limit, one obtains an action[18]

$$
\begin{aligned}
S \;=\; & -\frac{L}{4\pi}\int d\sigma dt\left(\sin^2\frac{\theta_2}{2}\dot{\chi}+\cos^2\frac{\theta_2}{2}\cos\theta_1\dot{\phi}\right) \\
& -\frac{\lambda}{16\pi L}\int d\sigma dt\left({\theta_2'}^2+\cos^2\frac{\theta_2}{2}\left({\theta_1'}^2+\sin^2\theta_1{\phi'}^2\right)+\sin^2\frac{\theta_2}{2}\cos^2\frac{\theta_2}{2}\left(\chi'-\cos\theta_1\phi'\right)^2\right),
\end{aligned}
\tag{4.6}
$$

where the dot and prime denote the derivative of t and σ, respectively. The SU(2) action of the previous section is obtain by $\theta_2 = 0$ and $\theta_1 = \theta$ here. A periodic condition for $|\vec{n}\rangle$ imposes boundary conditions for $\phi(\sigma),\ \chi(\sigma)$,

$$
\phi(\sigma+2\pi) \;=\; \phi(\sigma)-2\pi\frac{k(p-q)}{M}, \tag{4.7}
$$

$$
\chi(\sigma+2\pi) \;=\; \chi(\sigma)+2\pi\frac{k}{M}(p+q-2r)\,. \tag{4.8}
$$

The translation condition (4.4) means

$$
P \equiv \frac{2\pi}{L}\int d\sigma T_{01} = -\frac{1}{2}\int d\sigma(\sin^2\frac{\theta_2}{2}\chi'+\cos^2\frac{\theta_2}{2}\cos\theta_1\phi') = -2\pi\frac{k(p+q)}{2M}\,. \tag{4.9}
$$

Next, consider the string side. In the SU(3) sector, a string rotates in $R_t \times S^5$. The relevant part of the metric is

$$
ds^2 = -dt^2 + d\Omega_5^2 = -dt^2 + d\theta^2 + \sin^2\theta d\phi_3^2 + \cos^2\theta(d\psi^2 + \cos^2\psi d\phi_1^2 + \sin^2\psi d\phi_2^2)\,. \tag{4.10}
$$

A, B and C is expressed by these coordinates as $A = \cos\theta\cos\psi e^{i\phi_1}$, $B = \cos\theta\sin\psi e^{i\phi_2}$, $C = \sin\theta e^{i\phi_3}$. The identification by orbifolding is expressed as $(\phi_1,\ \phi_2,\ \phi_3) \sim (\phi_1 + 2\pi\frac{p}{M},\ \phi_2 + 2\pi\frac{q}{M},\ \phi_3 + 2\pi\frac{r}{M})$.

Introducing next coordinates

$$
\phi_1 = \varphi_1 + \varphi_2\,, \qquad \phi_2 = \varphi_1 - \varphi_2\,, \qquad \phi_3 = \varphi_1 + \alpha\,, \tag{4.11}
$$

then the metric becomes

$$
\begin{aligned}
ds^2 \;=\; & -dt^2 + d\theta^2 + \sin^2\theta d\alpha^2 + 2\sin^2\theta d\alpha d\varphi_1 + d\varphi_1^2 + \\
& + \cos^2\theta(d\psi^2 + 2\cos(2\psi)d\varphi_2 d\varphi_1 + d\varphi_2^2)\,.
\end{aligned}
\tag{4.12}
$$

In the new coordinate, the string coordinates have the following boundary conditions,

$$
\varphi_1(\sigma+2\pi) \;=\; \varphi_1(\sigma) + 2\pi\frac{k(p+q)}{2M}\,, \tag{4.13}
$$

$$
\varphi_2(\sigma+2\pi) \;=\; \varphi_2(\sigma) + 2\pi\frac{k(p-q)}{2M}\,, \tag{4.14}
$$

$$
\alpha(\sigma+2\pi) \;=\; \alpha(\sigma) + 2\pi\frac{k}{M}(r - \frac{p+q}{2})\,. \tag{4.15}
$$

One consider a string with large angular momentum along to φ_1 in this background. Subtracting this fast motion from the Polyakov action in (4.12), the action becomes the same as that of the spin chain (4.6) with identifying $\theta = \frac{\theta_2}{2}$, $\psi = \frac{\theta_1}{2}$, $\varphi_2 = -\frac{\phi}{2}$, $\alpha = -\frac{\chi}{2}$ [18].

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The Virasoro constraint imposes

$$\int d\sigma(\cos^2\theta\cos(2\psi)\varphi_2' + \sin^2\theta\alpha') = -\int d\sigma\varphi_1' = -\frac{2\pi}{2}\left(m_1 + m_2 + \frac{k(p+q)}{M}\right), \quad (4.16)$$

where we use (4.13). Remembering $\theta = \frac{\theta_2}{2}$, $\psi = \frac{\theta_1}{2}$, $\varphi_2 = -\frac{\phi}{2}$, $\alpha = -\frac{\chi}{2}$, the equations (4.14), (4.15) and (4.16) are equivalent to the equations (4.7), (4.8) and (4.9).

5. Conclusions

In this paper, we have shown the agreements, at one-loop level of field theory, between the energies of the semiclassical strings in $AdS_5 \times S^5/\mathbb{Z}_M$ and the anomalous dimensions of the long length operators of $\mathcal{N} = 0, 1, 2$ planar orbifold field theories originating from $\mathcal{N} = 4$ SYM. On the gauge theory side, we have found that the orbifold field theories have the integrabilities in SU(2) and SU(3) sectors even when these symmetries are broken by the orbifold action. For an untwisted sector of operators, a corresponding spin chain is a SU(2)(SU(3)) XXX spin chain with periodic boundary condition. For twisted sectors of operators, if this SU(2)(SU(3)) is global symmetry of the orbifold theory, corresponding spin chain models are also XXX spin chains with periodic boundary condition. On the other hand, in the case of the broken symmetry, spin chains corresponding to twisted sectors of operators become XXX spin chains with twisted boundary condition. These facts suggest that $SU(2,2|4)$ super spin chains with twisted boundary condition correspond to anomalous dimension matrices of full-sector operators in orbifold theories. This may be the case even for the non-supersymmetric orbifold field theories.

With the spin chains corresponding to the orbifold theories in hand, we compared the 2-spin circular string solutions of the strings in $AdS_5 \times S^5/\mathbb{Z}_M$ to the rational solutions of the SU(2) XXX spin chain. We saw the agreement between the energies of the strings and the anomalous dimensions of the operators at one-loop level. Especially, the strings in the twisted sectors correspond to the 'twisted sectors' of the operators. Furthermore, we have generalized the agreements between the sigma models of string and gauge sides to the orbifold cases. This is done for SU(2) sectors and SU(3) sectors.

Finally, let us briefly discuss higher loops. Also at higher loops, the phase shifts born at interactions which step γ_k are cancelled by the same twisted boundary condition as at one-loop level. Then, corresponding spin chain is the same as that in $\mathcal{N} = 4$ case [25, 26] except for twisted boundary condition. Hence, the agreements of string energies and anomalous dimensions would hold in higher loop level, if these hold in $\mathcal{N} = 4$ case. It is known that this agreement survive at 2-loop level, but there are discrepancy at 3-loop level in $\mathcal{N} = 4$ case [25, 26]. So, this would be the case for orbifold cases we consider.

It would be interesting to expand our results to more broaden class of operators, say full-sector of operators.

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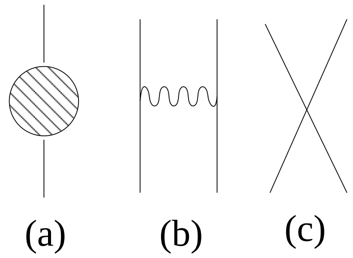

Figure 3: Diagrams which contribute to Z-factor.

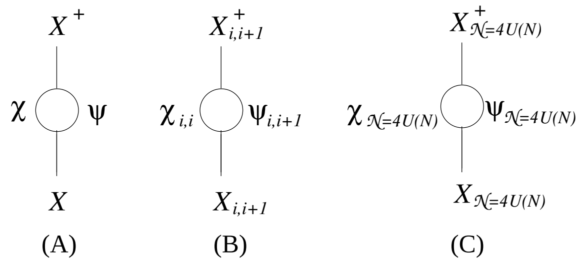

Figure 4: Diagram from yukawa coupling $\mathrm{tr}X^\dagger\chi\psi$. A diagram (A) consist of $NM \times NM$ matrix valued fields (2.21) in orbifold theory. χ is gaugino and ψ is $\mathcal{N}=1$ superpartner of X. This diagram (A) includes only one diagram (B) which is concerned with a component field $X_{i,i+1}$, which is an $N \times N$ matrix. Suffix of field $X_{i,i+1}$, $\chi_{i,i}$, $\psi_{i,i+1}$ represent gauge group 'name'. A diagram (C) is a corresponding diagram of $\mathcal{N}=4$ U(N) SYM.

A. Anomalous dimension matrix in untwisted sector of operators

We consider untwisted operators consisting of X, Y fields in $\mathcal{N}=2$ case. These are $NM \times NM$ matrix valued fields (2.21). The operators we consider have the form of $\mathrm{tr}XXY\cdots$, etc. Arguments in this section can be applied also to SU(2)(SU(3)) subsectors of section 3(section 4).

There are three types of diagrams figure 3 which contribute to renormalization Z-factor. Vertices of orbifold theory are inherited from $\mathcal{N}=4$ U(NM) SYM. Hence, there is one to one correspondence between diagrams of orbifold theory and diagrams of $\mathcal{N}=4$ SYM. Furthermore, contribution of each diagram to Z-factor in orbifold theory agrees with that in $\mathcal{N}=4$ U(N) SYM. Let us explain the last statement.

It is convenient to see diagrams in terms of component fields, which is $N \times N$ matrix valued (X_{12}, Y_{21}, X_{45}, etc). For instance, in a diagram consisting of $NM \times NM$ matrix valued fields X, χ and ψ of the orbifold theory (figure 4(A)), there is only one diagram which is concerned with a field $X_{i,i+1}$ (figure 4(B)). In figure 4(B), fields which run a loop are bifundamental fermions which are $N \times N$ matrices. Let us compare the diagram figure 4(B) with a diagram of $\mathcal{N}=4$ U(N) SYM (figure 4(C)). In figure 4(C), fields which run a loop are adjoint fermions which are $N \times N$ matrices and then, the value of this diagram is the same as that of the diagram of figure 4(B). This fact is true for each $N \times N$ valued component fields. Since the diagram figure 4(A) is merely linear summation of the diagram figure 4(B) for each component field, a contribution of figure 4(A) to Z-factor in orbifold theory is the same as that of figure 4(C) to Z-factor in $\mathcal{N}=4$ U(N) SYM. This fact is approved for other diagrams in figure 3. Therefore a form of anomalous dimension matrix of untwisted operators in orbifold theory is the same as that of $\mathcal{N}=4$ U(N) SYM.

B. Rational solutions of Bethe equations

Here, we outline how to solve Bethe equations and translation condition in large L limit and obtain anomalous dimension using the solutions. We use a method which are described in [43, appendix]. For details, we would like to refer to the original paper.

The equations we consider are

$$\frac{1}{x_i} = \frac{2}{L} \sum_{j=1, j\neq i}^{J} \frac{1}{x_i - x_j} - 2\pi n, \tag{B.1}$$

$$\frac{1}{L} \sum_{i=1}^{J} \frac{1}{x_i} = -2\pi m\,, \tag{B.2}$$

where m and n are not necessarily integers. Let us define the resolvent

$$G(x) = \frac{1}{L} \sum_{j=1}^{J} \frac{1}{x - x_j}\,. \tag{B.3}$$

The total momentum and total energy, which is proportional to anomalous dimension of spin chain are described in terms of the resolvent

$$P = -G(0)\,, \qquad \gamma = -\frac{\lambda}{8\pi^2 L} G'(0)\,. \tag{B.4}$$

Multiplying (B.1) by $\frac{1}{x-x_i}$ and summing over i and using (B.2) appropriately, we obtain

$$xG^2(x) = G(x) + 2\pi n x G(x) - 2\pi m\,, \tag{B.5}$$

where we drop a term of subleading in $1/L$ expansion. This is quadratic equation of $G(x)$ and solution is

$$G(x) = \frac{1}{2x}\left(1 + 2\pi n x - \sqrt{(1+2\pi n x)^2 - 8\pi m x}\right). \tag{B.6}$$

This $G(x)$ goes to $\frac{m}{nx}$ in $x \to \infty$. This value should be equal to $\frac{J}{Lx}$. Hence, $\frac{J}{L} = \frac{m}{n}$ is imposed. So, m must have the same sign as n. Furthermore, Bethe ansatz assume $\frac{J}{L} \leq \frac{1}{2}$, so this imposes $|n| \geq 2|m|$. From these facts, $(n-m)$ has the same sign as m and n. Anyway, using the solution, we can obtain the anomalous dimension,

$$\gamma = \frac{\lambda}{2L} m(n-m)\,. \tag{B.7}$$

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